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Extending Cycles in Bipartite Graphs

GEORGE R. T. HENDRY

Department of Mathematics, Aberdeen University, Aberdeen AB9 2TY, Scotland

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Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n$. We introduce the following definitions. A cycle C in G is *extendable* if there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 2$. G is *bi-cycle extendable* if G has at least one cycle and every nonhamiltonian cycle in G is extendable. G has a *bipancyclic ordering* if the vertices of X and Y can be labelled $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$, respectively, so that $C_{2k} \subseteq \langle x_1, ..., x_k, y_1, ..., y_k \rangle$, for $2 \leqslant k \leqslant n$. Let

$$\bar{\sigma}(G) = \min\{d(x) + d(y) : x \in X, y \in Y, \text{ and } xy \notin E(G)\}.$$

It is shown that if $\bar{\sigma}(G) \geqslant n+1$ and C is a $2k$-cycle in G then C is extendable unless $\langle V(C) \rangle \cong K_{k,k}$. As consequences of the proof of this result, we deduce that if either $\bar{\sigma}(G) \geqslant (7n+1)/6$ or $\delta(G) \geqslant (n+1)/2$ then, in each case with one exceptional graph, G is bi-cycle extendable. It is also shown that if l is an integer such that $n \geqslant 2l \geqslant 2$, $\delta(G) \geqslant l$, and $|E(G)| \geqslant n^2 - ln + l^2$ then every cycle of length at least l in G is extendable unless $G \cong K_{n,n} - E(K_{l,n-l})$. As a corollary, we deduce that such a graph G has a bipancyclic ordering unless $G \cong K_{n,n} - E(K_{l,n-l})$. A number of preliminary results are required, among which is the determination of the maximum size of a balanced bipartite graph of specified order, minimum degree, and edge independence number. © 1991 Academic Press, Inc.

1. INTRODUCTION AND NOTATION

In [3], a cycle C in a graph G was said to be extendable if there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$, and it was shown to what extent certain known sufficient conditions for a graph to be hamiltonian imply the extendability of cycles. Our object in the present paper is to consider analogous questions relating to bipartite graphs. However, since no cycle in a bipartite graph is extendable in the above sense, the definitions of [3] must be modified:

DEFINITION. A cycle C in a bipartite graph G is *extendable* if there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 2$.

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DEFINITION. A bipartite graph is *bi-cycle extendable* if G contains at least one cycle and every nonhamiltonian cycle in G is extendable.

DEFINITION [5]. A bipartite graph G of order $2n$ is *bipancyclic* if G contains cycles of every even length k, $4 \leqslant k \leqslant 2n$.

DEFINITION. A bipartite graph $G(X, Y, E)$ of order $2n$ has a *bipancyclic ordering* if the vertices of X and Y can be labelled $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$, respectively, so that the induced subgraph of G with vertex set $\{x_1, ..., x_k, y_1, ..., y_k\}$ is hamiltonian, for $2 \leqslant k \leqslant n$.

It follows immediately from these definitions that if G is bi-cycle extendable and is not a cycle then G has a bipancyclic ordering and that G has a bipancyclic ordering $\Rightarrow G$ is bipancyclic $\Rightarrow G$ is hamiltonian.

In this paper, we consider to what extent the following two sufficient conditions for a bipartite graph to be bipancyclic imply the extendability of cycles. If $G(X, Y, E)$ is a bipartite graph, define

$$\bar{\sigma}(G) = \min\{d(x) + d(y) : x \in X,\ y \in Y, \text{ and } xy \notin E(G)\}.$$

THEOREM A. *If $G(X, Y, E)$ is a balanced bipartite graph of order $2n$ such that $\bar{\sigma}(G) \geqslant n+1$ then G is bipancyclic.*

THEOREM B. *Let n and l be integers with $n \geqslant 2l \geqslant 2$. If $G(X, Y, E)$ is a balanced bipartite of order $2n$ satisfying $\delta(G) \geqslant l$ and $|E(G)| > n^2 - ln + l^2$ then G is bipancyclic.*

The proof of Theorem A is an immediate consequence of the theorem in [7], and that of Theorem B follows from [6, Corollary 2] and [5, Theorem I]. Moon and Moser [6] had previously shown that the hypotheses of Theorems A and B imply hamiltonicity.

In Section 2, we prove some preliminary results on the distribution of edges in the vicinity of a nonextendable cycle which are used repeatedly in proving the main results of Sections 3 and 4. We also determine the maximum size of a balanced bipartite graph of specified order, minimum degree, and edge independence number. The main result (Theorem 3.1) of Section 3, which is a bipartite analogue of Theorem 2 of [3], implies that if G satisfies the hypothesis of Theorem A and C is a nonextendable cycle in G of length $2k$ then $\langle V(C) \rangle \cong K_{k,k}$. We conjecture that if G satisfies the hypothesis of Theorem A then G has a bipancyclic ordering unless n is odd and G is one exceptional graph. The main result (Theorem 4.1) of Section 4 shows that if G satisfies the hypothesis of Theorem B then every cycle in G of length at least l is extendable. As a corollary, we deduce that a graph satisfying the hypothesis of Theorem B has a bipancyclic ordering.

The following definition and theorem due to Bondy and Chvátal [2] is useful in Section 2 and 3.

DEFINITION. The *bipartite closure* of a balanced bipartite graph $G(X, Y, E)$ of order $2n$ is the bipartite graph obtained from G by recursively joining pairs of nonadjacent vertices $x \in X$ and $y \in Y$ with $d(x)+d(y) \geqslant n+1$ until no such pair remains.

THEOREM C. A balanced bipartite graph is hamiltonian if and only if its bipartite closure is hamiltonian.

On the whole, our notation is consistent with that of [1]. A *bipartite graph* G with bipartition $\{X, Y\}$ and edge set E is denoted by $G(X, Y, E)$. G is *balanced* if $|X| = |Y|$. If U and V are disjoint subsets of $X \cup Y$ then $q(U, V)$ denotes the number of edges in G with one end in U and the other in V and $\langle U \rangle$ is the induced subgraph of G with vertex set U. If w is a vertex of G and F is a subgraph of G or a subset of $V(G)$ then $N_F(w)$ denotes the set, and $d_F(w)$ the number, of neighbours of w in F. We abbreviate $N_G(w)$ and $d_G(w)$ to $N(w)$ and $d(\omega)$, respectively. We also write $G-F$ instead of $\langle V(G)-V(F) \rangle$. An independent set of edges in G is called a *matching*. A matching of maximum cardinality in G is a *maximum matching*. The cardinality of a maximum matching in G is the *edge independence number* of G and is denoted by $\beta_1(G)$. A matching of cardinality t is a *t-matching*.

In the diagrams accompanying the text, a pair of parallel lines connecting two sets of vertices indicates that each vertex in one set is adjacent to each vertex in the other set.

2. PRELIMINARY RESULTS

In this section, we first prove two lemmas concerning the distribution of edges in the vicinity of a nonextendable cycle in a bipartite graph. Then we determine the maximum size of a balanced bipartite graph of specified order, minimum degree, and edge independence number. This result plays a part in the proof of Theorem 4.1.

We begin with some notation which is taken as standard throughout this section and in much of the sequel:

Notation. Suppose that C is a nonextendable cycle in a balanced bipartite graph $G(X, Y, E)$ of order $2n$. Let the vertices of C be cyclically labelled $x_1, y_1, x_2, y_2, ..., x_k, y_k$, where $2 \leqslant k \leqslant n-1$ and subscripts are taken modulo k. Let $V_1 = V(C)$, $V_2 = V(G) - V_1$, $F = \langle V_1 \rangle$, and $H = \langle V_2 \rangle$. Let $X_1 = \{x_i : 1 \leqslant i \leqslant k\} = X \cap V_1$, $Y_1 = \{y_i : 1 \leqslant i \leqslant k\} = Y \cap V_1$, $X_2 = X \cap V_2$

and $Y_2 = Y \cap V_2$. Let $x \in X_2$ and $y \in Y_2$ and denote by F' the graph $\langle V_1 \cup \{x, y\} \rangle$. Observe that, since C is nonextendable, F' is a nonhamiltonian balanced bipartite graph of order $2(k+1)$. If $A \subseteq V_1$, we denote by A^+ the set $\{y_i : x_i \in A\} \cup \{x_{i+1} : y_i \in A\}$. Suppose that $\{v_1, v_2, v_3, v_4\} \subseteq V_1$. We write $v_1 < v_2 < v_3 < v_4 < v_1$ if, starting at v_1 and traversing C in the direction $\cdots x_k y_k x_1 y_1 \cdots$, the vertices v_1, v_2, v_3, and v_4 are encountered in that order.

LEMMA 2.1. 1. *$|N_F(y)^+ \cap N_F(x)| \leqslant 1$ and $|N_F(x)^+ \cap N_F(y)| \leqslant 1$.*

2. *$d_F(x) + d_F(y) \leqslant k+1$. Furthermore, if $d_F(x) + d_F(y) = k+1$ then there exist (not necessarily distinct) integers $\alpha, \beta \in \{1, ..., k\}$ such that $N_F(x) = \{y_\alpha, y_{\alpha+1}, ..., y_\beta\}$ and $N_F(y) = \{x_{\beta+1}, x_{\beta+2}, ..., x_\alpha\}$.*

3. *If x_{j_1}, $x_{j_2} \in N_F(x)^+$* and *y_{i_1}, $y_{i_2} \in N_F(y)^+$, where either $x_{j_1} < x_{j_2} < y_{i_1} < y_{i_2} < x_{j_1}$ or $x_{j_1} < y_{i_2} < x_{j_2} < y_{i_1} < x_{j_1}$, then either $d_{F'}(x_{j_1}) + d_{F'}(y_{i_1}) \leqslant k+1$ or $d_{F'}(x_{j_2}) + d_{F'}(y_{i_2}) \leqslant k+1$.*

4. *If $d_F(x) \geqslant s$ and $d_F(y) \geqslant s$, where $s \geqslant 1$, then $|E(F)| \leqslant (s-1)(k+1) + (k-s+1)^2$.*

Proof. 1. Straightforward.

2. If $d_F(x) \leqslant 1$ then we are done. Therefore suppose that $d_F(x) \geqslant 2$. If y_i, $y_j \in N_F(x)$ then, by Lemma 2.1.1, x_i and x_j cannot both be in $N_F(y)$. Therefore $d_F(y) \leqslant k - (d_F(x) - 1)$ and so $d_F(x) + d_F(y) \leqslant k+1$. Now suppose that $d_F(x) + d_F(y) = k+1$. W.l.o.g. we may assume that xy_k, $yx_k \in E(G)$ and, for $1 \leqslant i \leqslant k-1$, G contains exactly one of xy_i and yx_i. By Lemma 2.1.1, G contains at most one of xy_{k-1} and yx_1. W.l.o.g. suppose that $yx_1 \notin E(G)$. Therefore $xy_1 \in E(G)$. If $d_F(x) = k$, then the result follows with $\alpha = k$ and $\beta = k-1$. Therefore suppose that there exists g, $1 \leqslant g \leqslant k-2$, such that x is adjacent to $y_k, y_1, ..., y_g$ but not to y_{g+1}. So $yx_{g+1} \in E(G)$ but y is not adjacent to $x_1, x_2, ..., x_g$. If y is adjacent to all of $x_{g+1}, x_{g+2}, ..., x_k$, then the result follows with $\alpha = k$ and $\beta = g$. Therefore suppose there exists h, $g+3 \leqslant h \leqslant k$, such that y is adjacent to $x_h, x_{h+1}, ..., x_k$ but not to x_{h-1}. Therefore $xy_{h-1} \in E(G)$ and C can be extended to the cycle $y_g x y_{h-1} x_{h-1} \cdots x_{g+1} y x_h y_h \cdots y_g$, which is a contradiction.

3. By Theorem C, the bipartite closure of F' is nonhamiltonian. If Lemma 2.1.3 is false then $F'' = F' + x_{j_1} y_{i_1} + x_{j_2} y_{i_2}$ is a subgraph of the bipartite closure of F'. Since it is easily verified that F'' is hamiltonian, we have a contradiction.

4. As the result is clearly true if $s = 1$, suppose $s \geqslant 2$. Suppose $A = \{x_{j_1}, ..., x_{j_s}\} \subseteq N_F(x)^+$ and $B = \{y_{i_1}, ..., y_{i_s}\} \subseteq N_F(y)^+$. By repeatedly applying Lemma 2.1.3 to the quadruples x_{j_h}, $x_{j_{h+1}}$, y_{i_h}, and $y_{i_{h+1}}$, $1 \leqslant h \leqslant$

$s-1$, and relabelling the vertices if necessary, we deduce that $d_F(x_{j_h})+d_F(y_{i_h})\leqslant k+1$, $1\leqslant h\leqslant s-1$. It follows that

$$|E(F)|\leqslant\sum_{h=1}^{s-1}(d_F(x_{j_h})+d_F(y_{i_h}))+q(X_1-(A-x_{j_s}),\ Y_1-(B-y_{i_s}))$$
$$\leqslant(s-1)(k+1)+(k-s+1)^2. \quad \blacksquare$$

If $xy\in E(G)$ then stronger conclusions can be drawn. The proofs of Lemma 2.2.1 and 2.2.3 are straightforward. As the proofs of Lemma 2.2.2 and 2.2.4 are similar to but shorter than those of Lemma 2.1.2 and 2.1.3, respectively, these are also omitted.

LEMMA 2.2. *Suppose that $xy\in E(G)$.*

1. *$N_F(y)^+\cap N_F(x)=\phi$ and $N_F(x)^+\cap N_F(y)=\phi$.*
2. *$d_F(x)+d_F(y)\leqslant k$. Furthermore, if $d_F(x)+d_F(y)=k$ then either $d_F(x)=k$ or $d_F(y)=k$.*
3. *If $xy_i\in E(G)$ then $d_F(x_i)+d_F(y)\leqslant k$ and $d_F(x_{i+1})+d_F(y)\leqslant k$.*
4. *If $x_j\in N_F(x)^+$ and $y_i\in N_F(y)^+$ then $d_F(x_j)+d_F(y_i)\leqslant k+1$.* $\blacksquare$

The following family of graphs provides the extremal graphs for the next lemma and for the results of Section 4.

Notation. For positive integers n_1 and n_4 and nonnegative integers n_2 and n_3, let $B(n_1,n_2,n_3,n_4)$ denote the bipartite graph defined by

$$V(B(n_1,n_2,n_3,n_4))=\bigcup_{i=1}^{4}W_i,\qquad \text{where } |W_i|=n_i,\ 1\leqslant i\leqslant 4,$$

and

$$E(B(n_1,n_2,n_3,n_4))=\bigcup_{i=1}^{3}\{uv\colon u\in W_i,\ v\in W_{i+1}\}.$$

LEMMA 2.3. *Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n$ with*

$$\delta(G)=d\geqslant 0 \tag{1}$$

and with edge independence number

$$\beta_1(G)=t\leqslant n-1. \tag{2}$$

Then $t\geqslant 2d$ and $|E(G)|\leqslant tn-d(t-d)$. Furthermore, $|E(G)|=tn-d(t-d)$ if and only if $G\cong B(n-t+d, d, t-d, n-d)$.

Proof. Let M be a t-matching in G. Define $f(M)$ by

$$f(M) = \sum_{v \in V(M)} d(v).$$

Let us assume that M is chosen so that if M' is any other t-matching in G, then $f(M) \leqslant f(M')$. Let $M = \{x_i y_i : 1 \leqslant i \leqslant t\}$, $X_1 = X - V(M)$, and $Y_1 = Y - V(M)$. Since $t \leqslant n-1$, $X_1 \neq \emptyset$ and $Y_1 \neq \emptyset$. Since $\beta_1(G) = t$,

$$G \text{ contains no } M\text{-alternating path,} \tag{3}$$

i.e., a path with one end vertex in X_1, the other end vertex in Y_1, and with every alternate edge in M. In particular, it follows that

$$q(X_1, Y_1) = 0. \tag{4}$$

Let $Y_2 = N(X_1)$, $X_3 = N(Y_1)$, $X_2 = \{x_i : y_i \in Y_2\}$, $Y_3 = \{y_i : x_i \in X_3\}$, $X_4 = X - \bigcup_{i=1}^{3} X_i$, and $Y_4 = Y - \bigcup_{i=1}^{3} Y_i$. We have

$$X_2 \cap X_3 = \emptyset, \tag{5}$$

for if $x_i \in X_2 \cap X_3$ then $y_i \in Y_2$ and there exist $x \in X_1$ and $y \in Y_1$ such that $x y_i x_i y$ is an M-alternating path, which contradicts (3). With $n_i = |X_i| = |Y_i|$, $1 \leqslant i \leqslant 4$, set, it follows from (5) that

$$n_1 + n_2 + n_3 + n_4 = n \tag{6}$$

and

$$n_2 + n_3 + n_4 = t. \tag{7}$$

We also have

$$q(X_2, Y_3) = 0, \tag{8}$$

for if $x_i \in X_2$ is adjacent to $y_j \in Y_3$ then there exist $x \in X_1$ and $y \in Y_1$ such that $x y_i x_i y_j x_j y$ is an M-alternating path, which contradicts (3). We may assume w.l.o.g. that

$$n_2 \leqslant n_3. \tag{9}$$

Therefore by (1), (7), (9), and the definition of Y_2, we have

$$2d = 2\delta(G) \leqslant 2n_2 \leqslant n_2 + n_3 \leqslant t, \tag{10}$$

which establishes the first required inequality.

Suppose $x_i \in X_2$. Then $y_i \in Y_2$ and there exists $x \in X_1$ adjacent to y_i. Let $M' = M - \{x_i y_i\} \cup \{xy_i\}$. Since M' is a t-matching in G, it follows from the choice of M that

$$f(M) \leqslant f(M') = f(M) - d(x_i) + d(x). \tag{11}$$

By (11) and the definition of Y_2, we have

$$d(x_i) \leqslant d(x) \leqslant n_2. \tag{12}$$

Now by (6), (7), (12), and the definitions of Y_2 and X_3, we have

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{4} \sum_{x \in X_i} d(x) \\ &\leqslant n_1 n_2 + n_2 n_2 + n_3(n_1 + n_2 + n_3 + n_4) + n_4(n_2 + n_3 + n_4) \\ &= tn + n_2^2 - tn_2 - n_1 n_4 \leqslant tn + n_2^2 - tn_2. \end{aligned} \tag{13}$$

By (10), $d \leqslant n_2 \leqslant t/2$. Therefore $n_2^2 - tn_2$ attains its maximum value when $n_2 = d$. Therefore by (13),

$$|E(G)| \leqslant tn + n_2^2 - tn_2 \leqslant tn - d(t-d), \tag{14}$$

which establishes the second required inequality.

Now suppose that $|E(G)| = tn - d(t-d)$. Then equality holds throughout (13) and (14) and so

$$n_2 = d, \tag{15}$$

$$d(x) = n_2, \qquad \text{for all} \quad x \in X_1 \cup X_2, \tag{16}$$

$$d(x) = n, \qquad \text{for all} \quad x \in X_3, \tag{17}$$

and

$$n_1 n_4 = 0. \tag{18}$$

By (2), (6), and (7), $n_1 = n - t \geqslant 1$ and so, by (18),

$$n_4 = 0. \tag{19}$$

It follows from (8), (15), (16), (17), and (19) and the definitions of Y_2 and X_3 that $G \cong B(n-t+d, d, t-d, n-d)$, where $W_1 = X_1 \cup X_2$, $W_2 = Y_2$, $W_3 = X_3$, and $W_4 = Y_1 \cup Y_3$. The proof is completed by observing that $B(n+t+d, d, t-d, n-d)$ has $2n$ vertices, $tn - d(t-d)$ edges, edge independence number t, and, since $t \geqslant 2d$, minimum degree d. ∎

COROLLARY 2.4. *Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n$ with $\beta_1(G) = t$. Then $|E(G)| \leqslant tn$ with equality if and only if $G \cong K_{t,n} \cup \bar{K}_{n-t}$, where $K_{n,n} \cup \bar{K}_0 \cong K_{n,n}$.*

Proof. If $t = n$ the result is obvious. If $t \leqslant n - 1$ then, since $K_{t,n} \cup \bar{K}_{n-t} \cong B(n-t, 0, t, n)$, the result follows from Lemma 2.3 with $d = 0$. ▮

3. ORE-TYPE RESULTS

In this section we obtain certain Ore-type sufficient conditions for the extendability of cycles in bipartite graphs. The main result (Theorem 3.1) implies that if G satisfies the hypothesis of Theorem A and C is a non-extendable cycle in G, then $\langle V(C) \rangle$ is a regular complete bipartite graph. This result extends Theorem A and is a bipartite analogue of Theorem 2 of [3]. We close the section by stating without proofs a number of corollaries of Theorem 3.1. But first we define two families of graphs which provide the extremal graphs for the results of this section.

Notation. Let $\mathscr{G}_{2n}$ (see Fig. 1) denote the set of balanced bipartite graphs $G(X, Y, E)$ of order $2n \geqslant 10$ satisfying the following conditions:

(i) $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, $V_1 = X_1 \cup Y_1$, $V_2 = X_2 \cup Y_2$, $X_2 = S_1 \cup S_2$, $Y_2 = R_1 \cup R_2$, $|X_1| = k$, $|S_1| = s$, $|S_2| = n-k-s$, $|Y_1| = k$, $|R_1| = r$, $|R_2| = n-k-r$,

(ii) $\bar{\sigma}(G) \geqslant n+1$,

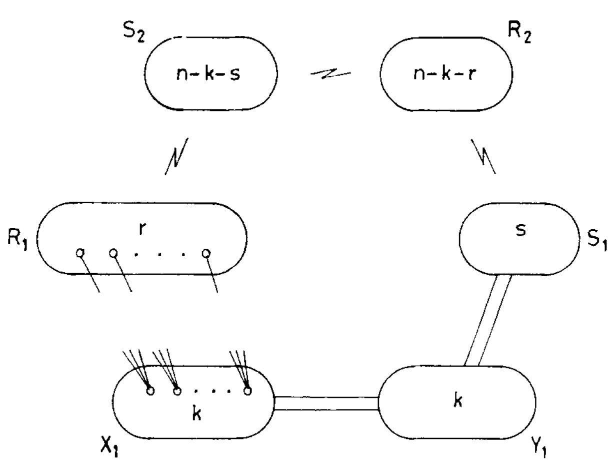

FIG. 1. The general structure of a graph belonging to $\mathscr{G}_{2n}$.

(iii) each vertex of G has at least one neighbour in V_2,

(iv) $d(v) \leqslant n-k$ for all $v \in V_2$,

(v) $q(X_1, R_2) = q(S_1, R_1) = q(S_2, Y_1) = 0$,

(vi) $r \geqslant k$, $s \geqslant 1$, $k \geqslant 2$, $n \geqslant r+s+k$, $s \leqslant \lfloor r/k \rfloor$,

(vii) $q(Y_1, X_1) = k^2$, $q(Y_1, S_1) = ks$,

(viii) each vertex of R_1 has exactly one neighbour in X_1, and

(ix) each vertex of S_2 has at least one neighbour in R_1.

Note that each graph in $\mathscr{G}_{2n}$ contains a nonextendable cycle of length $2k$ with vertex set $X_1 \cup Y_1$. The graphs $G_{2n,t}$ defined below belong to $\mathscr{G}_{2n}$.

Notation. For integers n and t, $1 \leqslant t \leqslant (n+1)/6$, such that $2t$ divides $n+1$, let $G_{2n,t}$ (see Figs. 2 and 3) denote the balanced bipartite graph of order $2n$ defined as follows:

(i) $V(G_{2n,t}) = X_1 \cup S_1 \cup S_2 \cup Y_1 \cup R_1 \cup R_2$, where $|X_1| = |Y_1| = (n+1)/2t - 1 = k$, $X_1 = \{x_1, ..., x_k\}$, $R_1 = \bigcup_{i=1}^{k} R_1^{(i)}$, $|S_1| = t$, $|S_2| = n-t-k$, $|R_2| = |R_1^{(i)}| = 2t-1$, $1 \leqslant i \leqslant k$, and

(ii)
$$E(G_{2n,t}) = \{xy: x \in X_1 \cup S_1,\ y \in Y_1\} \cup \{xy: x \in S_1 \cup S_2,\ y \in R_2\} \cup \{xy: x \in S_2,\ y \in R_1\} \cup \bigcup_{i=1}^{k} \{x_i y: y \in R_1^{(i)}\}.$$

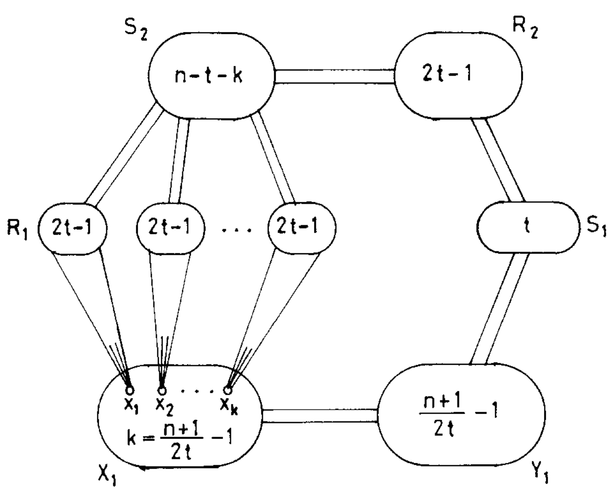

FIG. 2. The graph $G_{2n,t}$.

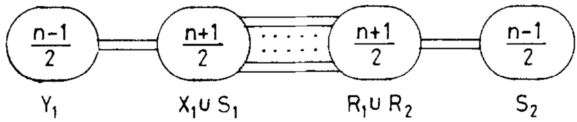

FIG. 3. The graph $G_{2n,1}$.

We now present the main result of this section:

THEOREM 3.1. *Let $G(X, Y, E)$ be balanced bipartite graph of order $2n \geq 4$ such that*

$$\bar{\sigma}(G) \geq n+1. \tag{1}$$

Then G is bi-cycle extendable unless $G \in \mathcal{G}_{2n}$. Furthermore, if C is a cycle of length $2k$, $2 \leq k \leq n-1$, in G and $\langle V(C) \rangle \not\cong K_{k,k}$ then C is extendable.

Proof. Suppose G is not bi-cycle extendable. By (1) and Theorem A, G contains at least one cycle. Therefore suppose that $C: x_1 y_1 x_2 y_2 \cdots x_k y_k x_1$ is a nonextendable cycle in G of length $2k$, where $2 \leq k \leq n-1$. We adopt the notation introduced at the beginning of Section 2. The proof consists of a series of propositions (2)–(17).

$$\delta(H) \geq 1. \tag{2}$$

Proof of (2). Suppose $x \in X_2$ and $d_H(x) = 0$. If $y \in Y_2$ then $xy \notin E(G)$, and so, by (1) and Lemma 2.1.2,

$$n+1 \leq d(x) + d(y) = (d_F(x) + d_F(y)) + d_H(y) \leq (k+1) + (n-k-1).$$

This contradiction establishes (2).

$$\text{If } v \in V_2 \text{ then } d(v) \leq n-k. \tag{3}$$

Proof of (3). W.l.o.g. suppose $v = x \in X_2$. By (2), there exists $y \in Y_2$ adjacent to x. If $d_F(x) = 0$ then we are done. So suppose that $d_F(x) \geq 1$ and that $w \in N_F(x)^+$. By Lemma 2.2.1, $yw \notin E(G)$. So by (1) and Lemma 2.2.3, we have

$$\begin{aligned} d_H(w) &= d(w) - d_F(w) \geq n+1 - d_H(y) - (d_F(y) + d_F(w)) \\ &\geq n+1-(n-k)-k = 1. \end{aligned} \tag{3.1}$$

Since C is not extendable, it follows from Lemmas 2.1.1 and 2.2.1 that no two vertices in $\{x\} \cup N_F(x)^+$ have a common neighbour in Y_2. Therefore, using (3.1), we deduce that

$$n-k=|Y_2| \geqslant d_H(x)+\sum_{w \in N_F(x)^+} d_H(w) \geqslant d_H(x)+d_F(x)=d(x).$$

$$\text{If } v \in V_1 \text{ then } d_H(v) \geqslant 1. \tag{4}$$

Proof of (4). Suppose that $d_H(x_1)=0$. If $y \in Y_2$ then, by (1) and (3), we have $n+1 \leqslant d(x_1)+d(y) \leqslant k+(n-k)$. This contradiction establishes (4).

If $x \in X_2$, $y \in Y_2$, and $xy \in E(G)$ then either $d_F(x)=0$ or $d_F(y)=0$. (5)

Proof of (5). Suppose that $d_F(x) \geqslant 1$ and $d_F(y) \geqslant 1$. Let $x' \in N_F(x)^+$ and $y' \in N_F(y)^+$. Then $F'+x'y$ and $F'+y'x$ are both hamiltonian. By Theorem C, we have

$$d_{F'}(x')+d_{F'}(y) \leqslant k+1 \qquad \text{and} \qquad d_{F'}(y')+d_{F'}(x) \leqslant k+1.$$

Since C is nonextendable,

$$d_{G-F'}(x')+d_{G-F'}(x) \leqslant n-k-1$$

and

$$d_{G-F'}(y')+d_{G-F'}(y) \leqslant n-k-1. \tag{5.1}$$

From (1) and these four inequalities, we have $2(n+1) \leqslant d(x')+d(y')+d(x)+d(y) \leqslant 2n$, which is a contradiction.

If $x \in X_2$, $y \in Y_2$, and $xy \notin E(G)$ then either $d_F(x) \leqslant 1$ or $d_F(y) \leqslant 1$. (6)

Proof of (6). Suppose that $d_F(x) \geqslant 2$ and $d_F(y) \geqslant 2$. By Lemma 2.1.3, there exist $x' \in N_F(x)^+$ and $y' \in N_F(y)^+$ such that

$$d_{F'}(x')+d_{F'}(y') \leqslant k+1. \tag{6.1}$$

Since (5.1) remains true, if

$$d_{F'}(x)+d_{F'}(y) \leqslant k+1 \tag{6.2}$$

then, as before, summing (5.1), (6.1), and (6.2) yields a contradiction. Hence we have $d_{F'}(x)+d_{F'}(y) \geqslant k+2$. Since $xy \notin E(G)$, this contradicts Lemma 2.1.2 and completes the proof of (6).

It follows from (5) and (6) that either $d_F(x) \leqslant 1$ for all $x \in X_2$ or $d_F(y) \leqslant 1$ for all $y \in Y_2$. W.l.o.g. we assume that

$$d_F(y) \leqslant 1 \qquad \text{for all} \quad y \in Y_2. \tag{7}$$

Define the sets R_i and S_i, $i = 1, 2$, as follows: $R_1 = N_H(X_1)$, $R_2 = Y_2 - R_1$, $S_2 = N_H(R_1)$, and $S_1 = X_2 - S_2$. Let $r = |R_1|$ and $s = |S_1|$. Then $|R_2| = n - k - r$ and $|S_2| = n - k - s$. (8)

$$q(X_1, R_2) = q(S_1, R_1) = q(S_2, Y_1) = 0. \tag{9}$$

Proof of (9). It is an immediate consequence of the definitions of R_1 and S_2 that $q(X_1, R_2) = q(S_1, R_1) = 0$. Suppose $x \in S_2$. By (8), there exist vertices $y \in R_1$ and $x' \in X_1$ such that xy, $yx' \in E(G)$. Then by (5), $d_F(x) = 0$ and hence $q(S_2, Y_1) = 0$. This establishes (9).

It follows by (7) and (8) that

$$\text{if } y \in R_1 \qquad \text{then} \quad d_F(y) = 1. \tag{10}$$

By (4), (9), and (10), we have

$$r \geqslant k \qquad \text{and} \qquad s \geqslant 1. \tag{11}$$

$$d(x) \leqslant \begin{cases} r+1 & \text{if} \quad x \in X_1 \\ n-r & \text{if} \quad x \in S_1 \\ n-k & \text{if} \quad x \in S_2. \end{cases} \tag{12}$$

$$d(y) \leqslant \begin{cases} k+s & \text{if} \quad y \in Y_1 \\ n-k-s+1 & \text{if} \quad y \in R_1 \\ n-k & \text{if} \quad y \in R_2. \end{cases} \tag{13}$$

Proof of (12) *and* (13). If $x \in S_1 \cup S_2$ or $y \in Y_1 \cup R_2$ then the result follows from (9). If $x \in X_1$ then by (4), (9), and (10), $d(x) = d_F(x) + d_{R_1}(x) \leqslant k + (r - (k-1)) = r + 1$. If $y \in R_1$ then by (9) and (10), $d(y) = d_F(y) + d_{S_2}(y) \leqslant 1 + n - k - s$.

$$n \geqslant r + s + k. \tag{14}$$

Proof of (14). If $x \in S_1$ and $y \in R_1$ then by (9), $xy \notin E(G)$. The result follows since, by (1), (12), and (13), we have $n + 1 \leqslant d(x) + d(y) \leqslant (n - r) + (n - k - s + 1)$.

$$s \leqslant \lfloor r/k \rfloor. \tag{15}$$

Proof of (15). By (9) and (10), we have $q(X_1, Y_2) = q(X_1, R_1) = r$. Therefore there exists $x \in X_1$ such that $d_H(x) \leqslant \lfloor r/k \rfloor$. Since $k \geqslant 2$, it follows by (4) and (10) that there exists $y \in R_1$ which is not adjacent to x. The result follows since, by (1) and (13), we have

$$n+1 \leqslant d(x)+d(y) \leqslant k+\lfloor r/k \rfloor+n-k-s+1.$$

$$q(Y_1, S_1)=ks. \tag{16}$$

Proof of (16). Suppose that there exist nonadjacent vertices $x \in S_1$ and $y \in Y_1$. Then by (1), (12), and (13), we have $n+1 \leqslant d(x)+d(y) \leqslant (n-r-1)+(k+s-1)$, and so $r-k-s+1 \leqslant -2$. Using (11) and (15), we deduce that $0 \leqslant (k-1)(s-1)=ks-k-s+1 \leqslant r-k-s+1 \leqslant -2$. This contradiction establishes (16).

$$F \cong K_{k,k}. \tag{17}$$

Proof of (17). Suppose there exist nonadjacent vertices $x \in X_1$ and $y \in Y_1$. Since $xy \notin E(G)$ and, by (9), $q(S_2, Y_1)=0$, we have $d(y) \leqslant k+s-1$. Therefore by (1), (12), and (14), $r+s+k+1 \leqslant n+1 \leqslant d(x)+d(y) \leqslant (r+1)+(k+s-1)$. This contradiction establishes (17).

It now follows from (1), (2), (3), (4), (8), (9), (10), (11), (14), (15), (16), and (17) that $G \in \mathscr{G}_{2n}$ and, by (17), that if C is a $2k$-cycle such that $\langle V(C) \rangle \not\cong K_{k,k}$ then C is extendable. ∎

Theorem 3.1 has a number of interesting consequences. We state these below without proof, as they all follow from Theorem 3.1 by routine arguments using the details of the structure of the graphs in $\mathscr{G}_{2n}$.

COROLLARY 3.2. *Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n \geqslant 4$ such that $\bar{\sigma}(G) \geqslant n+t$, where $t \geqslant 1$. Then every nonhamiltonian cycle in G of length at least $(n+1)/t-2$ is extendable unless $2t$ divides $n+1$ and $G \cong G_{2n,t}$.*

COROLLARY 3.3. *Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n \geqslant 4$ such that $\delta(G) \geqslant (n+1)/2$. Then G is bi-cycle extendable unless $n \geqslant 5$, n is odd, and $G \cong G_{2n,1}$.*

COROLLARY 3.4. *Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n \geqslant 4$ such that $\bar{\sigma}(G) \geqslant (7n+1)/6$. Then G is bi-cycle extendable unless $n+1$ is a multiple of 6 and $G \cong G_{2n,(n+1)/6}$.*

COROLLARY 3.5. *Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n \geqslant 8$ such that $\bar{\sigma}(G) \geqslant n+1$. If C is a cycle in G of length $2k$, $2 \leqslant k \leqslant n-2$, then there exists a cycle C'' in G of length $2k+4$ such that $V(C) \subseteq V(C'')$.*

Finally, we conjecture a further result:

Conjecture. Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n \geqslant 4$ such that $\bar{\sigma}(G) \geqslant n+1$. Then G has a bipancyclic ordering unless n is odd and $G \cong G_{2n,1}$.

4. Extremal Problems Involving Size and Minimum Degree

The results proved in this section are extensions of Theorem B. It is shown that if G satisfies the hypothesis of Theorem B then (Theorem 4.1) every cycle in G of length at least l is extendable and (Corollary 4.2) G has a bipancyclic ordering. Furthermore, there is a unique exceptional graph for each of these results.

Theorem 4.1. *Let n and l be integers with $n \geqslant 2l \geqslant 2$. Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n$ satisfying*

$$\delta(G) \geqslant l \tag{1}$$

and

$$|E(G)| \geqslant n^2 - ln + l^2. \tag{2}$$

Then every cycle in G of length at least l is extendable unless $G \cong B(l, l, n-l, n-l)$.

Proof. Suppose that $C\colon x_1 y_1 x_2 y_2 \cdots x_k y_k x_1$ is a nonextendable cycle in G of length $2k$, where

$$l \leqslant 2k \leqslant 2n-2. \tag{3}$$

We adopt the notation introduced at the beginning of Section 2. In addition, we assume that $X_2 = \{x_i : k+1 \leqslant i \leqslant n\}$, $Y_2 = \{y_i : k+1 \leqslant i \leqslant n\}$, $\beta_1(H) = t$, and w.l.o.g. that $M = \{x_i y_i : k+1 \leqslant i \leqslant k+t\}$ is a t-matching in H. By applying Corollary 2.4 to H, we deduce that

$$|E(H)| \leqslant t(n-k). \tag{4}$$

By applying Lemma 2.2.2 to the pair of vertices x_i and y_i, $k+1 \leqslant i \leqslant k+t$, and Lemma 2.1.2 to the pair of vertices x_i and y_i, $k+t+1 \leqslant i \leqslant n$, we deduce that

$$\begin{aligned} q(V_2, V_1) &= \sum_{i=k+1}^{n} (d_F(x_i) + d_F(y_i)) \\ &\leqslant tk + (n-k-t)(k+1). \end{aligned} \tag{5}$$

The remainder of the proof is divided into three propositions (6), (7), and (8).

$$\text{If } l \leqslant k \leqslant n-l \text{ then } G \cong B(l, l, n-l, n-l). \tag{6}$$

Proof of (6). Suppose $l \leqslant k \leqslant n-l$. Since $|E(F)| \leqslant k^2$, it follows from (2), (4), and (5) that

$$n^2 - ln + l^2 \leqslant |E(G)| \leqslant t(n-k) + tk + (n-k-t)(k+1) + k^2. \tag{6.1}$$

Therefore

$$k^2 - l^2 + ln - kn \geqslant (n-k-t)(n-k-1). \tag{6.2}$$

Since $t = \beta_1(H) \leqslant n-k$ and $k \leqslant n-1$, we have

$$(n-k-t)(n-k-1) \geqslant 0. \tag{6.3}$$

Therefore by (6.2) and (6.3),

$$(k-l)(k-n+l) \geqslant 0. \tag{6.4}$$

Since $l \leqslant k \leqslant n-l$, it follows from (6.4) that $k = l$ or $n-l$ and hence that equality holds throughout (6.1) and (6.3). Therefore

$$|E(H)| = t(n-k), \tag{6.5}$$

$$d_F(x_i) + d_F(y_i) = k, \qquad k+1 \leqslant i \leqslant k+t, \tag{6.6}$$

$$d_F(x_i) + d_F(y_i) = k+1, \qquad k+t+1 \leqslant i \leqslant n, \tag{6.7}$$

$$F \cong K_{k,k}, \tag{6.8}$$

and

$$\text{either } t = n-k \qquad \text{or} \qquad k = n-1. \tag{6.9}$$

First suppose $t = n-k$. By (6.5) and Corollary 2.4,

$$H \cong K_{n-k, n-k}. \tag{6.10}$$

By (6.6) and Lemma 2.2.2,

$$\text{either } d_F(x_i) = 0 \qquad \text{or} \qquad d_F(y_i) = 0, \quad k+1 \leqslant i \leqslant n. \tag{6.11}$$

Since C is not extendable, it follows w.l.o.g. from (6.6), (6.8), and (6.11) that $q(X_2, Y_1) = 0$ and $q(Y_2, X_1) = (n-k)k$. Since $k = l$ or $n-l$, it now follows from (6.8) and (6.10) that $G \cong B(l, l, n-l, n-l)$.

Now suppose $t \neq n-k$. By (6.9), $k = n-1$ and therefore $t = 0$. Since C is not extendable, it follows w.l.o.g. from (6.7) and (6.8) that $d_F(x_n) = k$ and $d_F(y_n) = 1$. Since $k = l$ or $n-l$ and $n \geqslant 2l$, it follows that $l = 1$ and hence that $G \cong B(l, l, n-l, n-l)$. This completes the proof of (6).

$$\text{If } n-l+1 \leqslant k \leqslant n-1 \text{ then } G \cong B(l, l, n-l, n-l). \tag{7}$$

Proof of (7). Suppose that

$$n-l+1 \leqslant k \leqslant n-1. \tag{7.1}$$

Therefore

$$l \geqslant 2. \tag{7.2}$$

We now consider two cases according to the value of t.

Case 1. Suppose $1 \leqslant t \leqslant n-k$. Since $t \geqslant 1$, $x_{k+1} y_{k+1} \in E(G)$. By (1) and (7.1), $d_F(x_{k+1}) \geqslant l-(n-k) \geqslant 1$ and $d_F(y_{k+1}) \geqslant l-(n-k) \geqslant 1$. W.l.o.g. suppose that $A = \{x_{j_h}: 1 \leqslant h \leqslant l-n+k\} \subseteq N_F(x_{k+1})^+$ and $B = \{y_{i_h}: 1 \leqslant h \leqslant l-n+k\} \subseteq N_F(y_{k+1})^+$. It follows by Lemma 2.2.3 that

$$d_F(y_{k+1}) + d_F(x_{j_1}) \leqslant k \qquad \text{and} \qquad d_F(x_{k+1}) + d_F(y_{i_1}) \leqslant k. \tag{7.3}$$

By Lemma 2.2.4, $d_F(x_{j_h}) + d_F(y_{i_h}) \leqslant k+1$, $2 \leqslant h \leqslant l-n+k$. It now follows from (2), (4), (7.3), and Lemmas 2.1.2 and 2.2.2 that

$$\begin{aligned} n^2 - ln + l^2 \leqslant |E(G)| &\leqslant |E(H)| + \sum_{i=k+2}^{n} (d_F(x_i) + d_F(y_i)) \\ &\quad + d_F(x_{k+1}) + d_F(y_{k+1}) + d_F(x_{j_1}) + d_F(y_{i_1}) \\ &\quad + \sum_{h=2}^{l-n+k} (d_F(x_{j_h}) + d_F(y_{i_h})) + q(X_1 - A, Y_1 - B) \\ &\leqslant t(n-k) + (t-1)k + (n-k-t)(k+1) + 2k \\ &\quad + (l-n+k-1)(k+1) + (k-(l-n+k))^2. \end{aligned}$$

Therefore $ln \leqslant tn + lk + l - kt - t - 1$, and so $l(n-k-1) < t(n-k-1)$. Since $n \geqslant k+1$, it follows that $l < t$. So by (7.1), $n-k+1 \leqslant l < t \leqslant n-k$. This contradiction means that Case 1 cannot occur.

Case 2. Suppose $t = 0$. Then by (1), each vertex in V_2 has at least l neighbours in V_1. Therefore by using (2), (4), (5), (7.2), and Lemma 2.1.4, we deduce that

$$\begin{aligned} n^2 - ln + l^2 \leqslant |E(G)| &\leqslant (n-k)(k+1) + (l-1)(k+1) \\ &\quad + (k-l+1)^2. \end{aligned} \tag{7.4}$$

Therefore $n(n-k-1) \leqslant l(n-k-1)$, and so, since $n \geqslant 2l$, $k=n-1$ and equality holds throughout (7.4). In particular,

$$d_F(x_n) + d_F(y_n) = k+1 \tag{7.5}$$

and

$$|E(F)| = (l-1)n + (n-l)^2. \tag{7.6}$$

By (7.5) and Lemma 2.1.2, we may assume w.l.o.g. that

$$N(y_n) = \{x_1, ..., x_\alpha\} \quad \text{and} \quad N(x_n) = \{y_\alpha, ..., y_{n-1}\}, \tag{7.7}$$

where, by (1) and (7.2),

$$2 \leqslant l \leqslant \alpha \leqslant n-l \leqslant n-2. \tag{7.8}$$

$$\text{For } 1 \leqslant i \leqslant \alpha - 1 \text{ and } \alpha + 1 \leqslant j \leqslant n-1,\ y_i x_j \notin E(F), \tag{7.9}$$

for otherwise, C can be extended to the cycle $y_i x_j y_j \cdots y_{n-1} x_n y_{j-1} x_n y_{j-1} x_{j-1} \cdots x_{i+1} y_n x_1 y_1 \cdots y_i$. Therefore $|E(F)| \leqslant (n-1)^2 - (\alpha-1)(n-1-\alpha)$. By combining this with (7.6), we deduce that $0 \leqslant (\alpha - l)(\alpha - n + l)$. Therefore, by (7.8), α equals l or $n-l$. In fact, we may assume w.l.o.g. that $\alpha = l$. Let $W_1 = \{y_n, y_1, ..., y_{\alpha-1}\}$, $W_2 = \{x_1, ..., x_\alpha\}$, $W_3 = \{y_\alpha, ..., y_{n-1}\}$, and $W_4 = \{x_{\alpha+1}, ..., x_n\}$. By (7.7) and (7.9), $q(W_1, W_4) = 0$. It therefore follows by (2) that $G \cong B(l, l, n-l, n-l)$, which establishes (7).

$$k \geqslant l. \tag{8}$$

Proof of (8). Suppose (8) is false. Then

$$k \leqslant l-1 \tag{8.1}$$

and so, since $k \geqslant 2$,

$$l \geqslant 3. \tag{8.2}$$

Suppose $0 \leqslant t \leqslant n-k-1$. It follows from (1) and (8.1) that $\delta(H) \geqslant l-k \geqslant 1$. Therefore by (2), (5), and Lemma 2.3, we deduce that

$$\begin{aligned} n^2 - ln + l^2 &\leqslant |E(H)| + q(V_2, V_1) + |E(F)| \\ &\leqslant t(n-k) - (l-k)(t-l+k) + tk + (n-k-t)(k+1) + k^2 \end{aligned}$$

and so $(n-t)(n-l-k-1) \leqslant k(k-2l+t-1)$. Since $t \leqslant n-k-1$ and $k \leqslant l-1$, it follows that $k(n-2l-2) \geqslant k(k-2l+t-1) \geqslant (n-t)(n-l-k-1) \geqslant (k+1)(n-2l)$, which is a contradiction. We therefore conclude that

$$t = n-k. \tag{8.3}$$

Let v be any vertex of V_2. We claim that

$$d_H(v) \leqslant n-k-(l-k)\, d_F(v). \tag{8.4}$$

Suppose w.l.o.g. that $v = y \in Y_2$. If $d_F(y)=0$ then (8.4) is clearly true. Therefore suppose that $d_F(y) \geqslant 1$. Let $A = N_F(y)^+$. By Lemma 2.1.1., no two vertices of A have a common neighbour in X_2 and, by Lemma 2.2.1, y is adjacent to no vertex of $N_H(A)$. It follows from these remarks and (1) that $d_H(y) \leqslant |X_2| - |N_H(A)| \leqslant n-k-(l-k)|A| = n-k-(l-k)\, d_F(y)$, which establishes (8.4). By summing (8.4) over all vertices $v \in V_2$, we obtain

$$2\,|E(H)| \leqslant 2(n-k)^2 - (l-k)\, q(V_2, V_1). \tag{8.5}$$

If $k = l-1$ then by using (2), (5), (8.3), and (8.5), we deduce that

$$\begin{aligned} 2n^2 - 2ln + 2l^2 &\leqslant (2\,|E(H)| + q(V_2, V_1)) + q(V_2, V_1) + 2\,|E(F)| \\ &\leqslant 2(n-l+1)^2 + (n-l+1)(l-1) + 2(l-1)^2. \end{aligned}$$

Consequently $(l-3)n \leqslant l^2 - 6l + 3$. Since $n \geqslant 2l$ and, by (8.2), $l \geqslant 3$, it follows that $2l^2 - 6l \leqslant l^2 - 6l + 3$. Since this is false for $l \geqslant 3$, we deduce that $k \neq l-1$. Therefore by (3) and (8.1),

$$l/2 \leqslant k \leqslant l-2. \tag{8.6}$$

It follows from (1) that

$$q(V_1, V_2) = \sum_{v \in V_1} d_H(v) \geqslant (l-k)|V_1| = 2(l-k)k. \tag{8.7}$$

Therefore by (2), (8.5), and (8.7), we have

$$\begin{aligned} n^2 - ln + l^2 &\leqslant |E(H)| + q(V_2, V_1) + |E(F)| \\ &\leqslant (n-k)^2 - \tfrac{1}{2}(l-k-2)\, q(V_2, V_1) + k^2 \\ &\leqslant n^2 - 2kn + 2k^2 - (l-k-2)(l-k)k \end{aligned}$$

and hence

$$n(2k-l) \leqslant (l-k)(k-l)(k+1) + k^2. \tag{8.8}$$

Since $n \geqslant 2l$, it follows from (8.6) that $n \geqslant 2l \geqslant 2(k+2)$ and $2k-l \geqslant 0$. Therefore by using (8.8), we deduce that $2(k+1)(2k-l) \leqslant n(2k-l) \leqslant k^2 - (k+1)(l-k)^2 < (k+1)^2 - (k+1)(l-k)^2$ and hence that

$$(l-k)^2 < k+1-2(2k-l) = 3(l-k)+1-l. \tag{8.9}$$

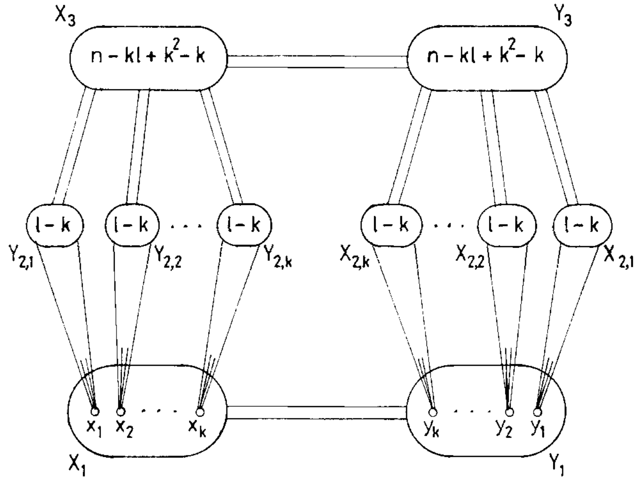

FIG. 4. The graph $A(n, k, l)$.

Since, by (8.6), $l-k \geqslant 2$, it follows that (8.9) is false. This contradiction establishes (8).

Statements (3), (6), (7), and (8) complete the proof of Theorem 4.1. ∎

The conclusion of Theorem 4.1 is best possible in the sense that a graph satisfying the hypothesis of Theorem 4.1 may contain nonextendable cycles of any length $2k$ less than l. To see this, for integers n, k, and l with $l > 2k \geqslant 4$ and n "large," define the bipartite graph $A(n, k, l)$ of order $2n$ (see Fig. 4) as

$$V(A(n, k, l)) = \bigcup_{i=1}^{3} (X_i \cup Y_i),$$

where $X_1 = \{x_1, ..., x_k\}$, $Y_1 = \{y_1, ..., y_k\}$, X_2 is the disjoint union of the sets $X_{2,i}$, $1 \leqslant i \leqslant k$, each of cardinality $l-k$, Y_2 is the disjoint union of the sets $Y_{2,i}$, $1 \leqslant i \leqslant k$, each of cardinality $l-k$, $|X_3| = |Y_3| = n - kl + k^2 - k$, and

$$\begin{aligned} E(A(n, k, l)) = {} & \{xy : x \in X_1,\ y \in Y_1\} \\ & \cup \bigcup_{i=1}^{k} (\{x_i y : y \in Y_{2,i}\} \cup \{y_i x : x \in X_{2,i}\}) \\ & \cup \{xy : x \in X_2,\ y \in Y_3\} \cup \{xy : x \in X_3,\ y \in Y_2 \cup Y_3\}. \end{aligned}$$

Provided n is sufficiently large in relation to k and l, it can be verified that $A(n, k, l)$ has minimum degree l and at least $n^2 - ln + l^2$ edges. However, any cycle in $A(n, k, l)$ of length $2k$ with vertex set $X_1 \cup Y_1$ is not extendable.

COROLLARY 4.2. *Let n and l be integers with $n \geqslant 2l \geqslant 2$. Let $G(X, Y, E)$ be a balanced bipartite graph of order $2n$ satisfying $\delta(G) \geqslant l$ and $|E(G)| \geqslant n^2 - ln + l^2$. Then G has a bipancyclic ordering unless $G \cong B(l, l, n-l, n-l)$.*

Proof. We proceed by induction on l. If $1 \leqslant l \leqslant 4$ then the result follows directly from Theorem 4.1. Therefore assume that n and l integers with $n \geqslant 2l \geqslant 10$ and that G is a balanced bipartite graph of order $2n$ satisfying

$$\delta(G) \geqslant l \tag{9}$$

and

$$|E(G)| \geqslant n^2 - ln + l^2. \tag{10}$$

Assume also that

$$\text{the result holds for smaller values of } l \tag{11}$$

but that

$$G \text{ does not have a bipancyclic ordering} \tag{12}$$

and

$$G \not\cong B(l, l, n-l, n-l). \tag{13}$$

First we show that

$$C_4 \subseteq G. \tag{14}$$

Proof of (14). Suppose to the contrary that $C_4 \nsubseteq G$. Since $l \leqslant n/2$, it follows from (10) that

$$|E(G)| \geqslant n^2 - \frac{n}{2} \cdot n + \left(\frac{n}{2}\right)^2 = \frac{3n^2}{4} \tag{14.1}$$

and hence that $\Delta(G) \geqslant 3n/4$. Suppose $x \in X$ with $d(x) = \Delta(G) = \Delta$. Since $C_4 \nsubseteq G$, each vertex of $X - \{x\}$ has at most one neighbour in $N(x)$. Therefore

$$\begin{aligned} |E(G)| &\leqslant \Delta + (n-1) + (n-1)(n-\Delta) = (2-n)\Delta + n^2 - 1 \\ &\leqslant (2-n)\frac{3n}{4} + n^2 - 1 \\ &= \frac{n^2}{4} + \frac{3n}{2} - 1 < \frac{3n^2}{4}. \end{aligned} \tag{14.2}$$

Since (14.1) and (14.2) are contradictory, (14) is established.

Let F be a balanced bipartite subgraph of G such that

$$F \text{ has a bipancyclic ordering} \tag{15}$$

and

$$\text{subject to (15), } |V(F)| \text{ is maximum.} \tag{16}$$

It follows from (14) that F exists. Let $V_1 = V(F)$, $V_2 = V(G) - V_1$, $|V_1| = 2k$, and $H = \langle V_2 \rangle$. By (15), there exists a hamiltonian cycle C in F and, by (16), C is not extendable in G. It follows by (13), (14), and Theorem 4.1 that

$$4 \leqslant 2k < l. \tag{17}$$

As in the proof of Theorem 4.1, Eq. (8), we deduce that H has a 1-factor. Therefore by Lemma 2.2.2, $q(V_2, V_1) \leqslant (n-k)k$. By (17), $|E(F)| \leqslant k^2 < lk$. Therefore, by (10),

$$\begin{aligned} |E(H)| &= |E(G)| - q(V_2, V_1) - |E(F)| \\ &> n^2 - ln + l^2 - (n-k)k - lk \\ &= (n-k)^2 - (l-k)(n-k) + (l-k)^2. \end{aligned} \tag{18}$$

Now $n-k$ and $l-k$ are integers with $n-k \geqslant 2l-k > 2(l-k)$ and, by (17), $l-k > k \geqslant 2$. Furthermore, H is a balanced bipartite graph of order $2(n-k)$ which, by (9) and (18), satisfies $\delta(H) \geqslant l-k$ and $|E(H)| > (n-k)^2 - (l-k)(n-k) + (l-k)^2$. Therefore, by the inductive hypothesis (11), H has a bipancyclic ordering. However, since, by (17), $|V(H)| = 2(n-k) > 2(l-k) > 2k = |V(F)|$, we have a contradiction of the choice of F (15, 16). It follows from this contradiction that either (12) or (13) is false. This completes the inductive step and hence the proof of Corollary 4.2. ∎

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References

1. M. Behzad, G. Chartrand, and L. Lesniak-Foster, "Graphs and Digraphs." Prindle, Weber & Schmidt, Boston, 1979.

2. J. A. BONDY AND V. CHVÁTAL, A method in graph theory. *Discrete Math.* **15** (1976), 111–135.
3. G. R. T. HENDRY, Extending cycles in graphs, *Discrete Math.* **85** (1990), 59–72.
4. G. R. T. HENDRY, "On Paths, Factors and Cycles in Graphs," Ph.D. Thesis, Aberdeen University, 1985.
5. J. MITCHEM AND E. SCHMEICHEL, Pancyclic graphs and bipancyclic graphs—A survey, preprint.
6. J. MOON AND L. MOSER, On hamiltonian bipartite graphs, *Israel J. Math.* **1** (1963), 163–165.
7. E. SCHMEICHEL AND J. MITCHEM, Bipartite graphs with cycles of all even lengths, *J. Graph Theory* **6** (1982), 429–439.